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Lagrangian velocity covariance in helical turbulence

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The effect of helicity on the Lagrangian velocity covariance $U^L(t)$ in isotropic, normally distributed turbulence is examined by computer simulation and by a renormalized perturbation expansion for $U^L(t)$. The first term of the latter represents Corrsin's (1959) conjecture (extrapolated to all t), which relates $U^L(t)$ to the Eulerian covariance and the distribution $G(\mathbf{x}, t)$ of fluid-element displacement. Truncation of the expansion at the first term yields the direct-interaction approximation for $G(x, t)$. The expansion suggests that with or without helicity Corrsin's conjecture is valid as $t \to \infty$ and that in either case $U^L(t)$ behaves asymptotically like $t^{-(r+\frac{3}{2})}$ if the spectrum of the Eulerian field varies like k^{r+2} at small wavenumbers. Corrsin's conjecture breaks down at small and moderate t if there is strong helicity while remaining accurate at all t in the mirror-symmetric case. Computer simulations for a frozen Eulerian field with spectrum confined to a thin spherical shell in k space indicate that strong helicity induces an increase in the Lagrangian correlation time by a factor of approximately three. Direct-interaction equations are constructed for the Lagrangian space-time covariance and the resulting prediction for $U^L(t)$ is compared with the simulations. The effect of helicity is well represented quantitatively by the direct-interaction equations for small and moderate t but not for large t. These frozen-field results imply good quantitative accuracy at all t in time-varying turbulence whose Eulerian correlation time is of the order of the eddy-circulation time. In turbulence with weak helicity, the direct-interaction equations imply that the Lagrangian correlation of vorticity with initial velocity is more persistent than $U^L(t)$, by a substantial factor.

1. Introduction

Some recent computer experiments on diffusion in homogeneous turbulence (Kraichnan 1976) have given the unexpected result that helicity in the turbulence enhances the diffusion of fluid elements. In the present paper, this phenomenon is studied analytically, and the computer experiments are extended to longer diffusion times. Our investigation is confined to statistically stationary, isotropic turbulence, for analytical simplicity, but the results carry over to the more general homogeneous case. We use the word isotropic here to imply rotational invariance and distinguish between reflexionally invariant and helical statistics within the isotropic category.

Let $u_i(\mathbf{x}, t)$ denote the Eulerian velocity field and consider the Lagrangian velocity

$$v_i(t) = u_i(\mathbf{y}(t), t) \tag{1.1}$$

of the fluid element whose trajectory

$$y_i(t) = \int_0^t v_i(s)\, ds \tag{1.2}$$

passes through $\mathbf{x} = 0$, $t = 0$. The Lagrangian velocity covariance and the diffusivity coefficient for fluid elements are given by

$$U^L(t) = \langle v_i(t)\, v_i(0)\rangle,$$
$$\kappa(t) = \frac{1}{3}\int_0^t U^L(s)\, ds. \tag{1.3}$$

An enhanced value of the steady-state diffusivity $\kappa(\infty)$ in helical turbulence implies that the Lagrangian velocity correlation is more persistent.

The effect of helicity on $U^L(t)$ can be straightforwardly displayed by repeatedly differentiating (1.1) at $t = 0$ and evaluating the Eulerian moments thus generated in order to determine the coefficients c_n in the expansion

$$U^L(t) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}. \tag{1.4}$$

The evaluation is simplest when the Eulerian field is frozen in time $[u_i(\mathbf{x}, t) = u_i(\mathbf{x})]$ and this case also gives the biggest difference between helical and reflexion-invariant diffusivities. Differentiation of (1.1) then gives

$$\left.\begin{aligned} dv_i/dt &= (\partial u_i/\partial x_j)\, v_j, \\ d^2v_i/dt^2 &= (\partial u_i/\partial x_j)\,(\partial u_j/\partial x_n)\, v_n + (\partial^2 u_i/\partial x_j\,\partial x_n)\, v_j\, v_n, \quad \text{etc.}\end{aligned}\right\} \tag{1.5}$$

Now assume that the Eulerian field is normally distributed. Multiply (1.5) by $v_i(t)$, take $t = 0$, and evaluate the ensemble averages of the resulting equations by the rules for a normal distribution. We find

$$\left.\begin{aligned} c_0 &= \langle u_i\, u_i\rangle, \quad c_1 = \langle(\partial u_i/\partial x_j)\, u_i\, u_j\rangle, \\ c_2 &= \langle(\partial u_i/\partial x_j)\,(\partial u_j/\partial x_n)\; u_n\, u_i\rangle + \langle(\partial^2 u_i/\partial x_j\,\partial x_n)\, u_j\, u_n\, u_i\rangle.\end{aligned}\right\} \tag{1.6}$$

The odd-order moments vanish, and the even-order moments are evaluated by taking all pairings. Those pairings which do not vanish because of homogeneity and incompressibility can be evaluated from the isotropic relations

$$\left.\begin{aligned} \langle u_j\, u_n\rangle = v_0^2\,\delta_{jn}, \quad \langle u_i(\partial u_j/\partial x_n)\rangle = \beta v_0^2\, k_\Omega\, \epsilon_{inj}, \\ \langle u_i\, \partial^2 u_i/\partial x_j\,\partial x_n)\rangle = -v_0^2\, k_\lambda^2\, \delta_{jn}.\end{aligned}\right\} \tag{1.7}$$

Here v_0 is the root-mean-square velocity in any direction and the characteristic wavenumbers k_Ω and k_λ are defined by

$$k_\Omega = \frac{2}{3v_0^2}\int_0^\infty E(k)\, k\, dk, \quad k_\lambda^2 = \frac{2}{3v_0^2}\int_0^\infty E(k)\, k^2\, dk, \tag{1.8}$$

where $E(k)$ is the usual turbulence energy spectrum per unit mass.

From (1.6) and (1.7),

$$c_0 = 3v_0^2, \quad c_1 = 0, \quad c_2 = -3v_0^4(k_\lambda^2 - 2\beta^2 k_\Omega^2). \tag{1.9}$$

Also, (1.7) implies that the helicity density is

$$\langle \mathbf{u}\,.\,\boldsymbol{\omega}\rangle = 6\beta v_0^2\, k_\Omega \tag{1.10}$$

and that the largest helicity which is kinematically possible is given by

Thus (1.9) shows that the curvature of $U^L(t)$ at $t = 0$ decreases with increasing helicity. In general, $k_\lambda \geqslant k_\Omega$, with equality when $E(k)$ has the form

$$E(k) = \tfrac{3}{2}v_0^2\,\delta(k-k_0). \tag{1.12}$$

In the latter case, $k_\lambda = k_\Omega = k_0$ and the curvature at $t = 0$ is reduced to half its reflexion-invariant value if the helicity is maximal.

Before proceeding with more analysis, it is appropriate to seek a simple physical explanation of the enhancement of diffusion by helicity. At first sight, the effect is puzzling because the Eulerian correlation $\langle u_i(\mathbf{x})\,u_i(\mathbf{x}')\rangle$ is determined by $E(k)$ alone and is independent of helicity. However, strong helicity decreases the probability of fluid-element trajectories which bend tightly back on themselves to return close to the origin. Such paths which lie nearly in a plane are strongly discriminated against. More generally, it is plausible that a path constrained to follow a helix cannot bend back on itself as tightly as one which is not, given a characteristic correlation length of the Eulerian field.

Equation (1.9) describes only the initial behaviour of $U^L(t)$, but it suggests that the effects of helicity on diffusivity are strongest when the energy spectrum is highly compact, so that $k_\Omega \sim k_\lambda$. It also seems clear that these effects should be strongest when the Eulerian velocity field is frozen, as assumed in obtaining (1.9). At the opposite extreme the correlation time of the Eulerian field is very short compared with an eddy-circulation time. In this case, the trajectory of a fluid element consists of a string of short, statistically independent straight-line segments whose statistics are independent of whether or not there is helicity.

2. Corrsin's conjecture and reflexion invariance

The probability distribution of the fluid-element displacement may be written as

$$G(\mathbf{x},t) = \langle \delta^3(\mathbf{x}-\mathbf{y}(t))\rangle. \tag{2.1}$$

Its transform

$$g(k,t) = \int G(\mathbf{x},t)\exp(-i\mathbf{k}\,.\,\mathbf{x})\,d^3x \tag{2.2}$$

is the mean response function of a Fourier mode of a passive scalar field convected by the turbulence. If k is very small compared with wavenumbers of the turbulence, Taylor's (1921) theory of diffusion implies

$$g(k,t) \approx \exp\left[-k^2\int_0^t \kappa(s)\,ds\right], \tag{2.3}$$

so that

$$U^L(t) = \{-3k^{-2}d^2[\ln g(k,t)]/dt^2\}_{k\to 0}. \tag{2.4}$$

Corrsin (1959) conjectured an asymptotic relation between $G(\mathbf{x},t)$ and $U^L(t)$ of a different kind. He started by noting that

$$U^L(t) = \int\langle u_i(0,0)\,u_i(\mathbf{x},t)\,\delta^3(\mathbf{x}-\mathbf{y}(t))\rangle\,d^3x \tag{2.5}$$

is an exact restatement of (1.3). Corrsin's conjecture is that for large t the statistical dependence of $\mathbf{u}(\mathbf{y}(t),t)$ on $\mathbf{y}(t)$ becomes sufficiently weak that the average in (2.5) can be factorized to yield the asymptotic equation

$$U^L(t) = \int \bar{U}_{ii}(\mathbf{x},t)\,G(\mathbf{x},t)\,d^3x, \tag{2.6}$$

where $\tilde{U}_{ij}(\mathbf{x},t) = \langle u_i(\mathbf{x},t)\,u_j(0,0)\rangle$. If

$$U(k,t) = U_{ii}(\mathbf{k},t), \quad U_{ij}(\mathbf{k},t) = (2\pi)^{-3}\int U_{ij}(\mathbf{x},t)\exp(-i\mathbf{k}.\mathbf{x})\,d^3x, \tag{2.7}$$

this can also be written as

$$U^L(t) = \int U(k,t)\,g(k,t)\,d^3k. \tag{2.8}$$

Equation (2.8) also arises as an analytical consequence of the direct-interaction approximation (DIA) for $g(k,t)$ (Roberts 1961). The DIA integral equation for $g(k,t)$ is

$$\partial g(k,t)/\partial t = -\int_0^t ds \int d^3p\, k_i k_j\, U_{ij}(\mathbf{k}-\mathbf{p},t-s)\,g(p,t-s)\,g(k,s), \tag{2.9}$$

and (2.8) follows from the solution together with (2.3) and (2.4).

It is hard to measure either $U^L(t)$ or $g(k,t)$ at large t, so that a direct test of Corrsin's equation as an asymptotic relation is difficult. However, the solution of (2.9), which implies (2.8) at *all* t, agrees very well with computer simulations of diffusion at small, intermediate and large times for the exceptionally critical case of a frozen Eulerian field with spectrum (1.12) and reflexion-invariant normal statistics (Kraichnan 1970*a*). This strongly suggests that, provided there is reflexion invariance, Corrsin's conjecture is better than an asymptotic approximation: it is a uniformly valid approximation for all t.

Now consider the effects of helicity. Both the computer simulations (Kraichnan 1976) and the initial behaviour (1.9) indicate that $U^L(t)$ falls off more slowly and $\kappa(t)$ rises more rapidly when there is strong helicity. The latter behaviour implies, in turn, that $g(k,t)$ falls off more rapidly. Thus if (2.8) is a good approximation for given $U(k,t)$ in the reflexion-invariant case it cannot be good in the maximally helical case, because the t dependences of the right and left sides of the equation change in opposite senses. In the case of a frozen Eulerian field with spectrum (1.12) the discrepancies in the curvature at $t = 0$ and in the effective decay rate at $v_0 k_0 t \sim 1$ are both of the order of a factor of two.

The direct-interaction equation (2.9) may be regarded as a truncation of an infinite integro-differential series expansion corresponding to a renormalization of the perturbation expansion for $g(k,t)$ (Kraichnan 1961, 1970*b*). The higher terms omitted in (2.9) have the form of a functional power series in ascending powers of U_{ij}, with increasing numbers of integrations over intermediate times and wave vectors. Inclusion of the higher terms adds corresponding terms to (2.8) resulting in a formally exact infinite-series representation of $U^L(t)$. Written out explicitly up to terms quadratic in U_{ij}, this expansion is†

$$U^L(t) = \int U(k,t)\,g(k,t)\,d^3k - \frac{1}{2}\int_0^t ds \int_0^s ds' \int d^3k \int d^3k' \times k'_j k_m\, U_{ij}(\mathbf{k},t-s')\,U_{mi}(\mathbf{k}',s)\,g(k,t-s)\,g(|\mathbf{k}+\mathbf{k}'|,s-s')\,g(k',s') + \ldots. \tag{2.10}$$

The general isotropic form for U_{ij} is

$$U_{ij}(\mathbf{k},t) = \tfrac{1}{2}P_{ij}(\mathbf{k})\,U(k,t) + i\epsilon_{imj}k_m X(k,t), \quad P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2, \tag{2.11}$$

† Cf. Kraichnan [1970*b*, equations (48)–(53)]. Each $P(\ )$ factor in these equations should be $\frac{1}{2}P(\ )$.

where the function $X(k,t)$ vanishes if there is reflexion invariance. The contribution of the second term in (2.10) to $[d^2U^L(t)/dt^2]_{t=0}$ is

$$-\tfrac{1}{2}\int d^3k \int d^3k' \, k'_j \, U_{ij}(\mathbf{k},0)\, U_{mi}(\mathbf{k}',0)\, k_m, \tag{2.12}$$

where we note that $g(k,0) = 1$. In the reflexion-invariant case, we see that the integrand of the k integration in (2.12) changes sign under $\mathbf{k}\rightarrow -\mathbf{k}$, so that the integral vanishes by symmetry. This is not the case if $X(k,0)$ is not zero, and the resulting contribution corresponds to the β^2 term in (1.9). The higher terms not explicitly shown in (2.10) all involve more than two time integrations and therefore make no contribution to the initial curvature of $U^L(t)$ whether or not there is reflexion invariance.

The vanishing of (2.12) in the reflexion-invariant case may be regarded as a bonus. In general, the DIA is exact only up to the second order of perturbation theory. In the present case, this means that, if the solution of (2.9) for $g(k,t)$ is expanded in powers of t, it must agree with the exact $g(k,t)$ up to the term in t^2. But by (2.4) this means that $U^L(t)$ is guaranteed to be accurate only up to the trivial c_0 term in (1.4). In the reflexion-invariant case only, the vanishing of (2.12) means an extra two orders of accuracy, so that the DIA result for $g(k,t)$ as $k\rightarrow 0$ is accurate up to the t^4 term and the corresponding $U^L(t)$ is accurate up to the t^2 term. Note that $g(k,t)$ at all k is independent of the existence of helicity, up to order t^2. The X term in (2.11) makes no contribution to (2.9) because of the antisymmetry of ϵ_{imj}.

The renormalized expansion (2.10) may also be used to examine the validity of (2.8) as $t\rightarrow\infty$, which is Corrsin's actual conjecture. Consider, first, frozen normal Eulerian fields with spectra of the form

$$U(k) \propto k^r f(k/k_0), \quad f(0) = 1, \tag{2.13}$$

where f falls off rapidly enough at large k that $U(k)$ is integrable over k space and where k_0 is now a characteristic wavenumber of the energy-containing range. Let v_0 continue to be the root-mean-square velocity in any direction so that a typical eddy-circulation time is $\tau_0 = 1/v_0 k_0$. If the diffusion does not show anomalies at long times, then by (2.3)

$$g(k,t) \sim \exp(-k^2\kappa(\infty)\,t) \quad (k \ll k_0, \quad t \gg \tau_0). \tag{2.14}$$

Use of (2.13) and (2.14) in (2.8) yields

$$U^L(t) \sim v_0^2(t/\tau_0)^{-\frac{1}{2}(r+3)} \quad (t \gg \tau_0) \tag{2.15}$$

and it is easy to verify that the dominant contribution to $U^L(t)$ for large t comes from $k \sim (\tau_0/t)^{\frac{1}{2}}k_0$.†

The second term in (2.10) gives two principal contributions to $U^L(t)$ for $t \gg \tau_0$, one in which both k and k' are of order $(\tau_0/t)^{\frac{1}{2}}k_0$ and another in which one of these wavenumbers is of that order and the other is of order k_0. Both contributions are easily estimated if (2.14) remains approximately valid for $k \sim k_0$. The result of this analysis is that both contributions fall off as a higher power of t^{-1} than (2.15), provided $r \geq 0$. It can also be seen that the higher terms in (2.10), those not shown explicitly, contribute successively higher powers of t^{-1}. Thus, subject to reservations because (2.10) is at best an asymptotic rather than a convergent series, the renormalized perturbation theory supports Corrsin's conjecture that (2.8) is asymptotically correct for large t, and this appears to be true whether or not there is helicity. The behaviour described holds

† Saffman (1962) noted long ago that Corrsin's conjecture implies an algebraic fall off of $U^L(t)$ at large t.

whether X in (2.11) is assumed to be zero or to have the maximum value kinematically possible,

$$k|X(k)| = \tfrac{1}{2}U(k). \tag{2.16}$$

If the Eulerian field is time dependent and $U(k,t)$ and $X(k,t)$ fall off strongly after a finite correlation time, then the dominance of the first term in (2.10) over successive terms increases at large t. In the extreme case where this correlation time is $\ll \tau_0$ for all k, the dominance of the first term extends to all t. Again, this is true whatever the strength of the helicity.

The intuitive argument in support of Corrsin's conjecture is that for $t \gg \tau_0$ a typical fluid element has executed a random walk of many steps, the probability distribution $G(\mathbf{x},t)$ is wide compared with the correlation length $1/k_0$, and the probability that a fluid element has wandered back to within a correlation length of its starting point is independent of its initial velocity. If statistical dependence in the Eulerian field falls rapidly to zero for separations larger than order $1/k_0$, this argument appears difficult to fault, whether or not there is helicity. Also, it is easy to estimate that those fluid elements which linger for the entire time t in the neighbourhood of their origin make a negligible correction for large t.

However, this intuitive argument is not obviously valid if the spectrum has, say, the form (1.12), so that spatial correlations fall off slowly. In this case, all the terms in (2.10) appear to give contributions of the same order of magnitude at large t, so that the renormalized perturbation expansion also does not support Corrsin's conjecture.

It is of interest to ask what happens to the asymptotic behaviour (2.15) when the Eulerian field shows time dependence appropriate to Navier–Stokes dynamics. If the Reynolds number $v_0/\nu k_0$ is moderate or large, the time dependence of $U(k,t)$ for $k \ll k_0$ and $t \gg \tau_0$ should reflect two principal effects: the eddy viscosity exerted by wavenumbers of order k_0 and the eddy circulation associated with motions of scale k^{-1}. The effective eddy viscosity should be $\sim \kappa(\infty)$, implying a time dependence for $U(k,t)$ like (2.14). With the spectrum (2.13), the characteristic velocity associated with scales $\sim k^{-1}$ is $v_k \sim v_0(k/k_0)^{\frac{1}{2}(r+3)}$, so that the eddy-circulation time is

$$1/v_k k \sim (k_0/k)^{\frac{1}{2}(r+5)}\tau_0,$$

which is larger than the eddy–viscous decay time $(k_0/k)^2\tau_0$ as $k \to 0$ provided that $r \geq 0$. The result is that the time dependence of $U(k,t)$ does not change the asymptotic behaviour (2.15).

If $r = 0$, (2.13) gives equipartition of energy among the small wavenumbers and the $t^{-\frac{3}{2}}$ tail on $U^L(t)$ resembles that due to fluctuations of a quiescent fluid about thermal equilibrium (Kraichnan 1975). It is also of interest to note here that, if the turbulence exhibits a Kolmogorov inertial range and t is such that a typical fluid element has wandered from its origin a distance that lies within the inertial range of scales, then the LHDI approximation (Kraichnan 1966) predicts that $U^L(t)$ falls off like t^{-2}, apart from a dependence of logarithmic type.

3. Direct-interaction approximation for the Lagrangian covariance

One way to construct an analytical approximation which can distinguish the differing behaviours of $U^L(t)$ in reflexion-invariant and helical turbulence is to form a higher-order closure equation for $g(k,t)$ accurate up to the fourth order of perturbation theory. This would give correctly the curvature of $U^L(t)$ at $t = 0$. A closure which accomplishes this formally involves vertex renormalization of the perturbation series (Kraichnan 1961, 1974). This approximation gives excellent results for $g(k,t)$ at all t if $k \gg k_0$, but its qualitative properties and internal consistency have not been demonstrated for the smaller k values of interest here. The approximation is analytically complicated.

Instead, we shall examine here a closure for the Lagrangian velocity covariance which stays within the framework of the direct-interaction approximation, and therefore agrees exactly with perturbation theory up to only second order, but which works directly with $U^L(t)$ so that that quantity itself is correct up to second order. This is accomplished by introducing the generalized velocity field $\mathbf{u}(\mathbf{x},t|r)$, defined as the velocity measured at time r in the fluid element whose trajectory passes through $(\mathbf{x},t)$. The generalized field is related to the Eulerian velocity by

$$\mathbf{u}(\mathbf{x},t|t) = \mathbf{u}(\mathbf{x},t) \tag{3.1}$$

and to the previously defined Lagrangian velocity by

$$\mathbf{v}(t) = \mathbf{u}(0,0|t). \tag{3.2}$$

The Eulerian field determines $\mathbf{u}(\mathbf{x},t|r)$ by (3.1) and the equation of motion

$$[\partial/\partial t + \mathbf{u}(\mathbf{x},t).\nabla]\, u_i(\mathbf{x},t|r) = 0 \tag{3.3}$$

(Kraichnan 1965).

The covariance $U^L(t)$ satisfies $U^L(t) = U^L(-t)$ if the turbulence is stationary and homogeneous, since it is then immaterial whether an ensemble of trajectories is taken with the same starting point or the same finishing point. We have

$$U^L(t-r) = \langle u_i(\mathbf{x},t|r)\, u_i(\mathbf{x},t)\rangle. \tag{3.4}$$

If $U^L_{ij}(\mathbf{k},t)$ is defined by

$$U^L_{ij}(\mathbf{k},t-r) = (2\pi)^{-3}\int \langle u_i(\mathbf{x},t|r)\, u_j(\mathbf{x}',t)\rangle \exp[-i\mathbf{k}.(\mathbf{x}-\mathbf{x}')]\, d^3x, \tag{3.5}$$

then

$$U^L(t) = \int U^L_{ii}(k,t)\, d^3k. \tag{3.6}$$

By (3.3), the three components of $u_i(\mathbf{x},t|r)$ are advected independently, with the result that $u_i(\mathbf{x},t|r)$ is in general not solenoidal for $t \neq r$. However $U^L_{ij}(\mathbf{k},t)$ is solenoidal in j and this, together with rotational invariance, requires the form

$$U^L_{ij}(\mathbf{k},t) = \tfrac{1}{2}P_{ij}(\mathbf{k})\, U^L(k,t) + i\epsilon_{imj}\, k_m\, X^L(k,t), \tag{3.7}$$

which is solenoidal in both i and j and similar in structure to (2.11). Thus $U^L_{ii}(\mathbf{k},t) = U^L(k,t)$. If $t = 0$, U^L_{ij} reduces to U_{ij}, so that

$$U^L(k,0) = U(k,0), \quad X^L(k,0) = X(k,0). \tag{3.8}$$

The DIA equations for the covariance of the generalized velocity can be developed from (3.3) in analogy with the analysis for a passive scalar, which obeys the same equation. The detailed algorithms for this are given in Kraichnan (1965), where they are later used to construct a modification of the DIA. Instead, we use here the straight DIA without modifications. The DIA equations give the evolution of the full covariance $\langle u_i(\mathbf{x},t|r)\, u_j(\mathbf{x}',t'|r')\rangle$, which has four time arguments. In order to compute the quantities of present interest, i.e. $\langle u_i(\mathbf{x},t|r)\, u_j(\mathbf{x}',t)\rangle$, it is necessary to deal with coupled equations involving the more general covariance $\langle u_i(\mathbf{x},t|r)\,u_j(\mathbf{x}',t')\rangle$, which has three time arguments of which only two are independent, because of stationarity. If the Eulerian field is time independent,

$$u_i(\mathbf{x},t|r) = u_i(\mathbf{x},t+\Delta t|r+\Delta t), \tag{3.9}$$

which results in a very great simplification, yielding coupled equations for $U^L(k,t)$ and $X^L(k,t)$ without involving the more general covariance.

The algorithms of Kraichnan (1965), together with the isotropic forms (2.11) and (3.7) and the use of (3.9), yield the pair of coupled equations

$$\partial U^L(k,t)/\partial t = -\pi\int_0^t ds \iint_\Delta (pq/k)\sin^2\theta\, g(p,t-s)\,[k^2U(q)\,U^L(k,s) + \tfrac{1}{2}kq\cos\theta\, U(k)\,U^L(q,s) - 2k^2q^2X(k)\,X^L(q,s)]\,dp\,dq, \tag{3.10}$$

$$\partial X^L(k,t)/\partial t = -\pi\int_0^t ds \iint_\Delta (pq/k)\sin^2\theta\, g(p,t-s)\,[k^2U(q)\,X^L(k,s) - \tfrac{1}{2}q^2U(k)\,X^L(q,s) + \tfrac{1}{2}kq\cos\theta\,X(k)\,U^L(q,s)]\,dp\,dq. \tag{3.11}$$

In these equations $\iint_\Delta$ denotes integration over all p and q which can form a triangle with k, θ is the interior angle between k and q in this triangle, and $g(p,t-s)$ is the function defined by (2.2). The equation for $g(k,t)$ is that obtained by substituting (2.11) into the previous DIA equation (2.9):

$$\partial g(k,t)/\partial t = -\pi k^2\int_0^t ds \iint_\Delta (pq/k)\sin^2\theta\, U(q)\,g(p,t-s)\,g(k,s)\,dp\,dq. \tag{3.12}$$

The function $g(k,t)$ arises in (3.10) and (3.11) as the mean response function for infinitesimal perturbations of the generalized velocity, whose components, as we have noted, obey the same equation as a passively advected scalar field. Equations (3.10)–(3.12) can be integrated forwards in time from the initial values (3.8) and $g(k,0) = 1$. Note that the right-hand side of (3.10) consists of terms which contain either two U factors or two X factors, while all the terms on the right-hand side of (3.11) have one U and one X factor.

The curvatures at $t = 0$ of $U^L(k,t)$, $X^L(k,t)$ and $g(k,t)$ obtained from (3.10)–(3.12) are all exactly correct. Since (3.10) and (3.12) are not identical equations, the solutions do not satisfy (2.8) when the Eulerian field is frozen and therefore do not satisfy it in general. This is true even if the turbulence is reflexion-invariant, so that X vanishes. Then (3.10) and (3.12) differ by the $\cos\theta$ term in (3.10). That term vanishes by symmetry on performing the p integration at $t = 0$ because $g(p,0)$ is independent of p.

Equations (3.10) and (3.11) can be simplified if the helicity spectrum is a constant fraction of the maximal value at all k:

$$X(k) = \beta k^{-1}U(k) \quad (|\beta| < \tfrac{1}{2}). \tag{3.13}$$

If $Q(k,t)$ and $R(k,t)$ are defined by

$$U^L(k,t) = Q(k,t)\,U(k), \quad X^L(k,t) = R(k,t)\,X(k), \tag{3.14}$$

(3.10) and (3.11) yield

$$\partial Q(k,t)/\partial t = -\pi \int_0^t ds \iint_\Delta (pq/k) \sin^2\theta\, g(p,t-s)\, U(q) \times [k^2 Q(k,s) - 2\beta^2 kqR(q,s) + \tfrac{1}{2}kq\cos\theta\, Q(q,s)]\, dp\, dq, \tag{3.15}$$

$$\partial R(k,t)/\partial t = -\pi \int_0^t ds \iint_\Delta (pq/k) \sin^2\theta\, g(p,t-s)\, U(q) \times [k^2 R(k,s) - \tfrac{1}{2}kqR(q,s) + \tfrac{1}{2}kq\cos\theta\, Q(q,s)]\, dp\, dq. \tag{3.16}$$

If $|\beta| = \frac{1}{2}$, the R and Q equations become the same, giving $R(k,t) = Q(k,t)$ and

$$\partial Q(k,t)/\partial t = -\pi \int_0^t ds \iint_\Delta (pq/k) \sin^2\theta\, g(p,t-s)\, U(q) \times [k^2 Q(k,s) - \tfrac{1}{2}kq(1-\cos\theta)\, Q(q,s)]\, dp\, dq. \tag{3.17}$$

The identity of $R(k,t)$ and $Q(k,t)$ in this case is a property of the exact dynamics also. The velocity field is made up exclusively of waves with helicity of one sign and $U^L(k,t)$ and $X^L(k,t)$ are not independent.

The function $X^L(x,t)$ can be used to compute the Lagrangian covariance of vorticity with velocity:

$$H^L(t) = \langle \boldsymbol{\omega}(\mathbf{y}(t),t) \, . \, \mathbf{u}(\mathbf{y}(0),0)\rangle = \langle \boldsymbol{\omega}(\mathbf{x},t) \, . \, \mathbf{u}(\mathbf{x},t|0)\rangle. \tag{3.18}$$

The definition of vorticity

$$\omega_i(\mathbf{x},t) = i\epsilon_{imj}\, \partial u_j(\mathbf{x},t)/\partial x_m \tag{3.19}$$

and (2.11) yield

$$H^L(t) = 2\int k^2 X^L(k,t)\, d^3k. \tag{3.20}$$

Thus the DIA equations determine $H^L(t)$ as well as $U^L(t)$. However, they do not determine the Lagrangian correlation of vorticity with itself. Equation (3.19) does not generalize to give $\omega_i(\mathbf{x},t|r)$ in terms of $u_j(\mathbf{x},t|r)$ because the co-ordinates $\mathbf{x}$ label trajectories at time t and are not a Cartesian system at times $r \neq t$. In order to compute general covariances involving vorticity by the DIA it is necessary to augment (3.2) by a parallel equation for the propagation of $\omega_i(x,t|r)$.

4. Numerical results

Computer simulations of $U^L(t)$, $\kappa(t)$ and $g(k,t)$ were obtained by averaging over an ensemble of numerically integrated trajectories (1.1). The spectrum (1.12) was simulated by a set of 20 wave vectors randomly distributed on the sphere $|k| = k_0$. The frozen Eulerian field was then synthesized along the trajectory. A fresh realization of the Eulerian field was taken for each trajectory. The method of constructing the Eulerian field and of integrating the trajectories and then constructing the desired averages has been described previously (Kraichnan 1970*a*, 1976). The present simulations included 20000 realizations integrated to $t = 16/v_0 k_0$ for each of the two cases $\beta = 0$ and $\beta = 0{\cdot}5$ (reflexion invariance and maximal helicity, respectively). The integration time step was $\Delta t = 0{\cdot}25/v_0 k_0$. The results were compared with numerical integrations of the DIA equations (3.12), (3.15) and (3.16).

We shall discuss first the simulation results, shown in figures 1–3. The curves for $U^L(t)$ in figure 1 show the difference in initial curvature predicted by (1.9). However,

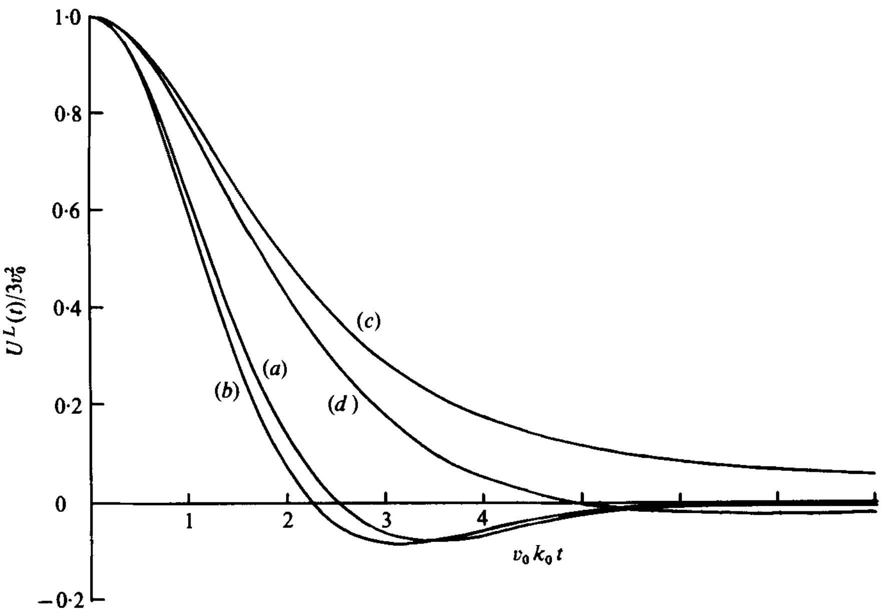

FIGURE 1. Lagrangian velocity covariance for frozen Eulerian field: (*a*) simulation for $\beta = 0$; (*b*) DIA for $\beta = 0$; (*c*) simulation for $\beta = \frac{1}{2}$; (*d*) DIA for $\beta = \frac{1}{2}$.

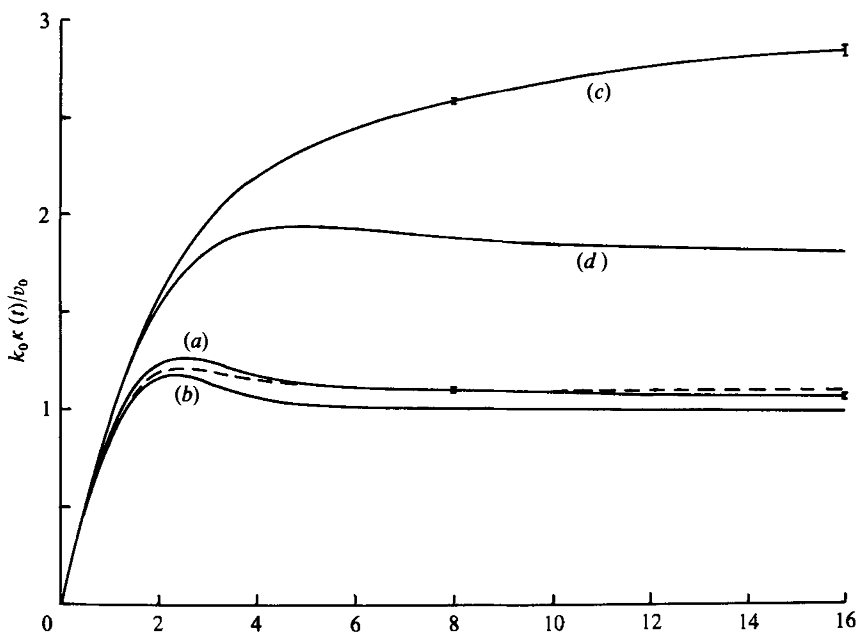

FIGURE 2. Diffusivity for frozen Eulerian field. Solid curves labelled as in figure 1. The dashed curve is obtained from the DIA $g(k, t)$ and (1.3) and (2.8).

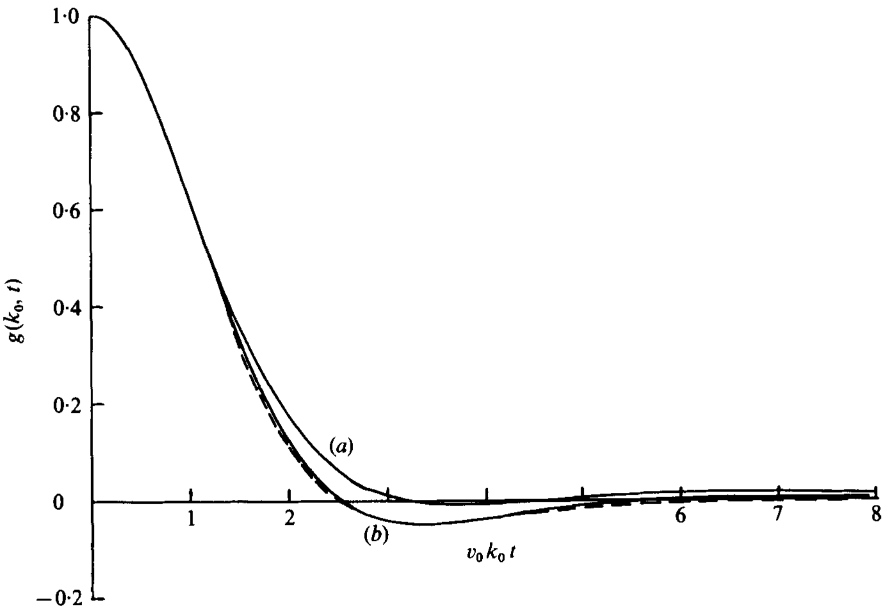

FIGURE 3. The function $g(k_0, t)$ for frozen Eulerian field: (a) simulation for $\beta = 0$; (b) simulation for $\beta = \frac{1}{2}$; ---, DIA.

the difference in $\kappa(\infty)$ between the helical and non-helical case is greater than would be expected from the initial behaviour. Equation (1.9) suggests that the characteristic time scale of $U^L(t)$ should increase by a factor $\sqrt{2}$ between $\beta = 0$ and $\beta = \frac{1}{2}$. But figure 2 shows that $\kappa(t)$, the time integral of $U^L(t)$, is higher for the helical case by a factor of 2·7 at $t = 16/v_0 k_0$ and that the ratio is still rising with t. It is not obvious from the figure that $\kappa(t)$ is actually approaching a finite limit as $t \to \infty$ in the helical case, but the numerical values for $U^L(t)$ at the larger t values do fall off faster than t^{-1}, thereby suggesting a finite limit.

Another difference in the behaviour of $U^L(t)$ in the two cases is that, for $\beta = 0$, $U^L(t)$ has a negative oscillation, reflecting the oscillatory spatial correlation function associated with (1.12). There is no trace of this behaviour for $\beta = \frac{1}{2}$.

The difference between the functions $\kappa(t)$ for the helical and non-helical cases implies a corresponding difference between the functions $g(k, t)$ for $k \ll k_0$, as discussed in § 2. Figure 3 shows the interesting fact that the degree of helicity has very little effect on $g(k, t)$ for $k = k_0$. For $k \gg k_0$, the fall-off of $g(k, t)$ is very close to what it would be in a spatially uniform velocity field, normally distributed, and therefore must asymptotically be completely independent of the presence of helicity as $k \to \infty$.

Standard deviations of the mean were computed for all the simulation results. The associated probable errors are too small to show on the plots except in the case of $\kappa(t)$, where they are indicated by bars.

The DIA values of the various functions are also shown on figures 1–3. The plots corroborate that the initial curvature of $U^L(t)$ is given correctly by the DIA equations. For $\beta = 0$, the agreement of the DIA function $U^L(t)$ with the simulation result is quite satisfactory for all t. In particular it reproduces the negative oscillation well. For $\beta = \frac{1}{2}$, the agreement is satisfactory for $t < 2/v_0 k_0$, but the DIA does not reproduce

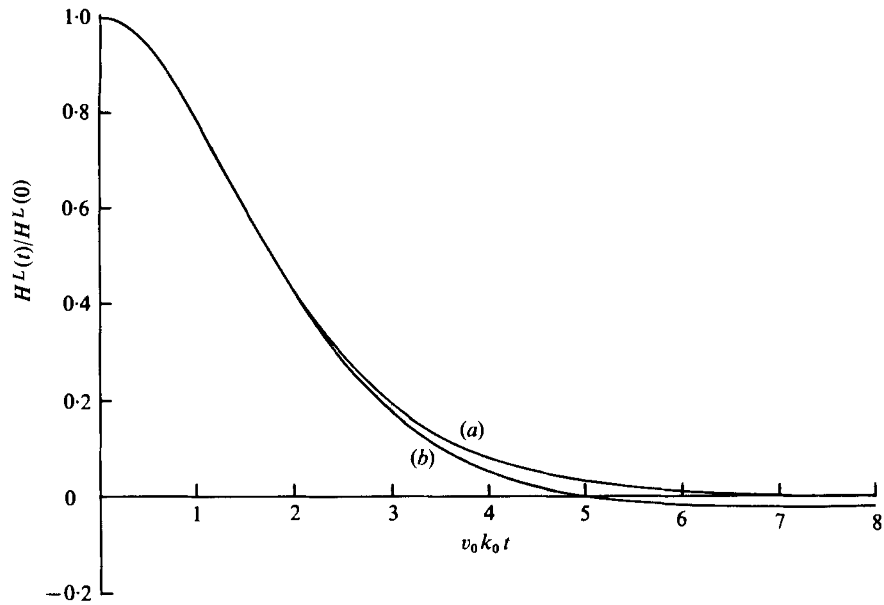

FIGURE 4. Lagrangian vorticity–velocity covariance for frozen Eulerian field according to DIA: (*a*) $\beta = 0$; (*b*) $\beta = \frac{1}{2}$.

well the slowly decaying tail for large t, and there is substantial error, whose relative size grows with t, for $t > 3/v_0 k_0$. The negative oscillation is much smaller than in the case $\beta = 0$ but is still there, while it is absent completely in the simulation curve. The integrated curves of figure 2 show the cumulative effect of the deviation of the DIA results from the simulation values. The faithfulness at all t for $\beta = 0$ and the deviation at large t for $\beta = \frac{1}{2}$ appear clearly. Also plotted in figure 2 is $\kappa(t)$ as computed from the DIA $g(k, t)$ used in (2.8). This shows an almost indecent agreement with the simulation values for $\beta = 0$. Figure 3 shows that the DIA $g(k_0, t)$ agrees well with the simulation results for $g(k_0, t)$ at both $\beta = 0$ and $\beta = \frac{1}{2}$.

Figure 4 shows the Lagrangian vorticity–velocity covariance $H^L(t)$ as computed from the DIA equations for both $\beta = 0$ and $\beta = \frac{1}{2}$. The two curves are close together. For $\beta = \frac{1}{2}$, $H^L(t)$ and $U^L(t)$ must be the same both in the DIA and in the exact dynamics, as we have discussed previously. For $\beta = 0$, however, the DIA results imply a substantially slower decay of $H^L(t)$ than of $U^L(t)$. This interesting result means that if weak helicity is present the correlation of vorticity with initial velocity is more persistent along particle trajectories than the correlation of the velocity itself. We have not attempted to corroborate this prediction by simulation because of the difficulty in getting good helicity statistics when the helicity is weak. The initial behaviour of $H^L(t)$ is of course guaranteed to be correct in the DIA equations.

The general conclusion from the comparisons of the DIA results with the simulations is that for the frozen velocity field with the spectrum chosen the DIA values for $g(k, t)$ are quite satisfactory for $k \sim k_0$ both with and without helicity. For $\beta = 0$, the DIA value for $g(k, t)$ is good also if $k \ll k_0$ but this is not so for $\beta = \frac{1}{2}$. In the latter case, good quantitative results for diffusion are obtained for moderate times if the DIA $g(k, t)$ is not used directly but instead is inserted in (3.15) and (3.16) to obtain $U^L(t)$ and the latter quantity is then used to obtain an improved approximation for $g(k,t)$ at small k by means of (2.3). After this procedure the DIA value of $\kappa(\infty)$ for $\beta = \frac{1}{2}$ is too small by a factor of over 1·5, but does exceed $\kappa(\infty)$ for $\beta = 0$ by substantially more than the factor $\sqrt{2}$ suggested by (1.9).

The frozen velocity field with spectrum (1.12), which we have used for the numerical integrations, probably gives the strongest differences between reflexion-invariant and helical diffusion and also probably maximizes the errors of the DIA. This is due to persistent correlations along typical trajectories both because the Eulerian field does not change in time and because the spatial correlation of the Eulerian field has a long-range tail which falls off like $\sin(kx)/(kx)$. If the Eulerian field instead has a finite correlation time τ_*, beyond which statistical correlations rapidly become negligible, then the long-range spatial tail no longer has an effect, because few particles move far enough to enter the tail region before the velocity field changes to a statistically independent form.

In the limit $\tau_* \ll \tau_0 \equiv 1/v_0 k_0$, the difference between helical and non-helical diffusion disappears, the diffusivity is $\kappa(\infty) = \tau_* v_0^2$ and the error of the DIA goes to zero (Kraichnan 1976). A more interesting and realistic choice is $\tau_* \sim \tau_0$. The shape of $\kappa(t)$ and the value of $\kappa(\infty)$ will then depend on the particular form of time correlation, space correlation (spectrum) and statistics taken for the velocity field. However, we can estimate both the magnitude of the changes in $\kappa(t)$ and $U^L(t)$ for finite τ_* and the effect on the accuracy of the DIA by using the present frozen-field results in a double-averaging procedure.

In its simplest form, the double-averaging procedure involves taking a frozen field over time intervals of duration $2\tau_*$, making the velocity field statistically independent, but with the same spectrum and β, on distinct time intervals, and staggering the transition times between intervals randomly from one subensemble of realizations to another. If either the DIA or the exact behaviour is computed for each subensemble and then averaged, the effect on the diffusivity is easily seen to be (Kraichnan 1976)

$$\kappa_*(t) = \begin{cases} (1-t/2\tau_*)\,\kappa(t) + (1/2\tau_*)\displaystyle\int_0^t \kappa(s)\,ds & (t \leqslant 2\tau_*), \\ \kappa_*(\infty) = (1/2\tau_*)\displaystyle\int_0^{2\tau_*} \kappa(s)\,ds & (t > 2\tau_*). \end{cases} \tag{4.1}$$

Here $\kappa_*(t)$ is the diffusivity for the double-averaged ensemble and $\kappa(t)$ is that for the frozen field. The Eulerian time correlation $\langle \mathbf{u}(\mathbf{x},t)\,.\,\mathbf{u}(\mathbf{x},t')\rangle$ falls linearly to zero at $t-t' = 2\tau_*$ and thus has the correlation time τ_*. The corresponding change in the Lagrangian correlation is

$$U_*^L(t) = \begin{cases} (1-t/2\tau_*)\,U^L(t) & (t \leqslant 2\tau_*), \\ 0 & (t > 2\tau_*). \end{cases} \tag{4.2}$$

Table 1 shows the values of $\kappa_*(\infty)$ for several values of τ_* for the exact dynamics and the DIA, with both $\beta = 0$ and $\beta = \frac{1}{2}$. For the exact dynamics, the ratio of diffusivity for $\beta = \frac{1}{2}$ to diffusivity for $\beta = 0$ rises from 1·134 at $\tau_* = \tau_0$ to 2·179 at $\tau_* = 8\tau_0$, while in the DIA this ratio goes from 1·145 to 1·741. The DIA gives the ratio with good accuracy for $\tau_* \leqslant 2\tau_0$ and gives the diffusivities themselves with an error of $\sim 6\,\%$ at $\tau_* = 2\tau_0$. Its accuracy deteriorates for larger τ_*.

τ_*/τ_0	Exact, $\beta = 0$	DIA, $\beta = 0$	Exact, $\beta = \frac{1}{2}$	DIA, $\beta = \frac{1}{2}$
1	0·780	0·758	0·885	0·869
2	1·008	0·943	1·415	1·326
4	1·066	0·982	1·920	1·623
8	1·072	0·990	2·335	1·723

TABLE 1. Values of $\kappa_*(\infty)$ for exact dynamics and DIA.

The frozen-field numerical results also yield values of $U^L(t)$ and $\kappa(t)$ for a generalized, more realistic, piecewise-continuous model in which the velocity field in each independent interval falls smoothly to zero at the ends of the interval, thereby giving a smooth Eulerian time correlation. This model can be handled by introducing an appropriate transformed time variable whose derivative is proportional to the form factor of the velocity field. It is unlikely that this refinement would appreciably affect either the dependence of the exact results on τ_* or the accuracy of the DIA results.

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REFERENCES

CORRSIN, S. 1959 *Atmospheric Diffusion and Air Pollution*, p. 161. Academic Press.
KRAICHNAN, R. H. 1961 *J. Math. Phys.* **2**, 124.
KRAICHNAN, R. H. 1965 *Phys. Fluids* **8**, 575.
KRAICHNAN, R. H. 1966 *Phys. Fluids* **9**, 1728.
KRAICHNAN, R. H. 1970*a* *Phys. Fluids* **13**, 22.
KRAICHNAN, R. H. 1970*b* In *The Padé Approximant in Theoretical Physics* (ed. G. A. Baker & J. L. Gammel), chap. 4. Academic Press.
KRAICHNAN, R. H. 1975 *Adv. in Math.* **16**, 305.
KRAICHNAN, R. H. 1976 *J. Fluid Mech.* **77**, 753.
ROBERTS, P. H. 1961 *J. Fluid Mech.* **11**, 257.
SAFFMAN, P. G. 1962 *Appl. Sci. Res.* A **11**, 245.
TAYLOR, G. I. 1921 *Proc. Lond. Math. Soc.* **20**, 196.